

The Syllogistic with Unity

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Abstract

We extend the language of the classical syllogisms with the sentence-forms “At most 1 p is a q ” and “More than 1 p is a q ”. We show that the resulting logic does not admit a finite set of syllogism-like rules whose associated derivation relation is sound and complete, even when *reductio ad absurdum* is allowed.

1 Introduction

By the *classical syllogistic*, we understand the set of English sentences of the forms

$$\begin{array}{ll} \text{No } p \text{ is a } q & \text{Some } p \text{ is a } q \\ \text{Every } p \text{ is a } q & \text{Some } p \text{ is not a } q, \end{array} \quad (1)$$

where p and q are common (count) nouns. By the *extended classical syllogistic*, we understand the classical syllogistic together with the set of quasi-English sentences of the forms

$$\text{Every non-}p \text{ is a } q \quad \text{Some non-}p \text{ is not a } q. \quad (2)$$

It is known that there exists a sound and complete proof system for the classical syllogistic in the form of a finite set of syllogism-like proof-rules (Smiley, 1973, Corcoran, 1972). Such a proof system also exists for the extended classical syllogistic; moreover, in both cases, *reductio ad absurdum*—in other words, the strategy of indirect proof—can be dispensed with (Pratt-Hartmann and Moss, 2009). The satisfiability problem for either of these languages is easily seen to be NLOGSPACE-complete, by a routine reduction to (and from) the problem 2-SAT.

Both the classical syllogistic and its extended variant may be equivalently reformulated using the numerical quantifiers “At most 0 ...” and “More than 0 ...”. The forms of the classical syllogistic thus become, respectively

$$\begin{array}{ll} \text{At most 0 } ps \text{ are } qs & \text{More than 0 } ps \text{ are } qs \\ \text{At most 0 } ps \text{ are not } qs & \text{More than 0 } ps \text{ are not } qs, \end{array} \quad (3)$$

while the additional forms of the extended classical syllogistic become

$$\text{At most } 0 \text{ non-}ps \text{ are not } qs \quad \text{More than } 0 \text{ non-}ps \text{ are not } qs. \quad (4)$$

These reformulations invite generalization. By the *numerical syllogistic*, we understand the set of English sentences of the forms

$$\begin{array}{ll} \text{At most } i \text{ } ps \text{ are } qs & \text{More than } i \text{ } ps \text{ are } qs \\ \text{At most } i \text{ } ps \text{ are not } qs & \text{More than } i \text{ } ps \text{ are not } qs, \end{array} \quad (5)$$

where p and q are common count nouns and i is a (decimal representation of a) non-negative integer. By the *extended numerical syllogistic*, we understand the numerical syllogistic together with the set of quasi-English sentences of the forms

$$\text{At most } i \text{ non-}ps \text{ are not } qs \quad \text{More than } i \text{ non-}ps \text{ are not } qs. \quad (6)$$

In other words, the classical syllogistic is simply the fragment of the numerical syllogistic in which all numbers are bounded by 0; and similarly for the extended variants. The first systematic investigation of the numerical syllogistic known to the author is that of De Morgan (1847, Ch. VIII), though this work was closely followed by treatments in Boole (1868) (reprinted as Boole, 1952, Sec. IV), and Jevons (1871) (reprinted as Jevons, 1890, Part I, Sec. IV). For a historical overview of this episode in logic, see Grattan-Guinness (2000). De Morgan presented a list of what he took to be the valid numerical syllogisms; and latter-day systems may be found in Hacker and Parry (1967) and Murphree (1998). It can be shown, however, that there exists *no* sound and complete syllogism-like proof system for the numerical syllogistic, even in the presence of *reductio ad absurdum*; and similarly for the extended numerical syllogistic (Pratt-Hartmann, 2009). In addition, the satisfiability problems for the numerical syllogistic and the extended numerical syllogistic are both NPTIME-complete (Pratt-Hartmann, 2008).

Thus, the numerical syllogistic differs from the classical syllogistic in its proof-theoretic and complexity-theoretic properties. The purpose of the present paper is to locate the source of this difference more precisely. Specifically, we consider the *syllogistic with unity*, which we take to consist of the classical syllogistic together with the forms

$$\begin{array}{ll} \text{At most } 1 \text{ } p \text{ is a } q & \text{More than } 1 \text{ } p \text{ is a } q \\ \text{At most } 1 \text{ } p \text{ is not a } q & \text{More than } 1 \text{ } p \text{ is not a } q, \end{array} \quad (7)$$

along with its extended variant, which additionally features the forms

$$\text{At most } 1 \text{ non-}p \text{ is not a } q \quad \text{More than } 1 \text{ non-}p \text{ is not a } q. \quad (8)$$

In other words, the syllogistic with unity is simply the fragment of the numerical syllogistic in which all numbers are bounded by 1; and similarly for the extended

variants. The syllogistic with unity gives rise to argument patterns having no counterparts in the classical syllogistic. For example,

$$\begin{array}{l}
\text{At most 1 } o \text{ is a } p \\
\text{At most 1 } o \text{ is not a } p \\
\text{At most 1 } q \text{ is not an } o \\
\hline
\text{More than 1 } q \text{ is not an } r \\
\hline
\text{At most 1 } q \text{ is an } r
\end{array} \tag{9}$$

is evidently a valid argument. For the first two premises ensure that there are at most two *os*, whence the third premise ensures that there are at most three *qs*; but the fourth premise states that at least two of these are not *rs*.

Syntactically speaking, the syllogistic with unity lies closer to the classical syllogistic than it does to the numerical syllogistic, because, like the former, but unlike the latter, it features only finitely many logical forms. This fact notwithstanding, we show in the sequel that there exists no sound and complete syllogism-like proof system for the syllogistic with unity, even in the presence of *reductio ad absurdum*; and similarly for its extended variant. We also observe that the satisfiability problem for either of these languages remains NP-TIME-complete. Thus, the smallest conceivable extension of the classical syllogistic by means of additional counting quantifiers yields the proof-theoretic and complexity-theoretic properties of the entire numerical syllogistic. Generalizing this result, we consider the family of languages obtained by restricting the numerical syllogistic so that all numbers are bounded by z , where z is any positive integer; and similarly for the extended numerical syllogistic. We show that, for all these languages, there exists no sound and complete syllogism-like proof system, even in the presence of *reductio ad absurdum*.

The syllogistic with unity exhibits some similarities with the intriguing logical system proposed by Hamilton (1860, pp. 249–317). According to Hamilton, the predicates of traditional syllogistic sentence-forms contain implicit existential quantifiers, so that, for example, “All p is q ” is to be understood as “All p is some q ”. Further, these implicit existential quantifiers can be meaningfully dualized to yield novel sentence-forms, thus: “All p is all q .” (A similar language was actually proposed in the earlier, but lesser-known Bentham, 1827). Hamilton’s account of the meanings of these sentences is, it must be said, unclear. However, a natural interpretation is obtained by taking the copula simply to denote the relation of identity. Thus, for example, the sentence “All p is all q ” is formalized by $\forall x(p(x) \rightarrow \forall y(q(y) \rightarrow x = y))$ —equivalently, either there are no ps , or there are no qs , or there is exactly 1 p and exactly 1 q , and they are identical (see, e.g. Fogelin, 1976). Under this interpretation, the pair of Hamiltonian sentences “All p are all p ” and “Some p are some p ” then states that there exists exactly one p —something assertable in the syllogistic with unity. In general, however, the two languages are expressively incomparable; in particular, the Hamiltonian syllogistic provides no means of stating that exactly one p is a q (with p and q different). Moreover, they exhibit different proof-theoretic properties: unlike the syllogistic with unity, Hamilton’s language does

indeed have a sound and complete syllogistic proof-system, though some form of indirect proof is essential (Pratt-Hartmann, 2011).

2 Syntax and semantics

Fix a countably infinite set \mathbf{P} . We refer to any element of \mathbf{P} as an *atom*. A *literal* is an expression of either of the forms p or \bar{p} , where p is an atom. A literal which is an atom is called *positive*, otherwise, *negative*. If $\ell = \bar{p}$ is a negative literal, then we denote by $\bar{\ell}$ the positive literal p . If z is a non-negative integer, an \mathcal{S}_z -*formula* is any expression of the forms

$$\exists_{\leq i}(p, \ell) \quad \exists_{> i}(p, \ell), \quad (10)$$

where p is an atom, ℓ is a literal and $0 \leq i \leq z$. An \mathcal{S}_z^\dagger -*formula* is any expression of the forms

$$\exists_{\leq i}(\ell, m) \quad \exists_{> i}(\ell, m), \quad (11)$$

where ℓ and m are literals and $0 \leq i \leq z$. We denote the set of \mathcal{S}_z -formulas simply by \mathcal{S}_z , and similarly for \mathcal{S}_z^\dagger . Where the language is clear from context, we speak simply of *formulas*. Evidently: $\mathcal{S}_z \subseteq \mathcal{S}_z^\dagger$, $\mathcal{S}_z \subseteq \mathcal{S}_{z+1}$, and $\mathcal{S}_z^\dagger \subseteq \mathcal{S}_{z+1}^\dagger$. We denote the union of all the languages \mathcal{S}_z by \mathcal{N} , and the union of all the languages \mathcal{S}_z^\dagger by \mathcal{N}^\dagger .

A *structure* is a pair $\mathfrak{A} = \langle A, \{p^\mathfrak{A}\}_{p \in \mathbf{P}} \rangle$, where A is a non-empty set, and $p^\mathfrak{A} \subseteq A$, for every $p \in \mathbf{P}$. The set A is called the *domain* of \mathfrak{A} . We extend the map $p \mapsto p^\mathfrak{A}$ to negative literals by setting, for any atom p ,

$$\bar{p}^\mathfrak{A} = A \setminus p^\mathfrak{A}.$$

Intuitively, we may think of the elements of \mathbf{P} as common count-nouns, such as “pacifist”, “quaker”, “republican”, etc., and if $a \in \ell^\mathfrak{A}$, we say that a *satisfies* ℓ in \mathfrak{A} , and regard a as having the property denoted by ℓ . Thus, we may gloss any negative literal \bar{p} as “non- p ” or “not a p ” depending on grammatical context. If \mathfrak{A} is a structure, we write $\mathfrak{A} \models \exists_{\leq i}(\ell, m)$ if $|\ell^\mathfrak{A} \cap m^\mathfrak{A}| \leq i$, and $\mathfrak{A} \models \exists_{> i}(\ell, m)$ if $|\ell^\mathfrak{A} \cap m^\mathfrak{A}| > i$. If $\mathfrak{A} \models \varphi$, we say that φ is *true* in the structure \mathfrak{A} . Thus, we may gloss $\exists_{\leq i}(\ell, m)$ as “At most i ℓ s are m s”, and $\exists_{> i}(\ell, m)$ as “More than i ℓ s are m ”. If $i > 0$, we write $\exists_{=i}(\ell, m)$ as an abbreviation for the pair of formulas $\{\exists_{>(i-1)}(\ell, m), \exists_{\leq i}(\ell, m)\}$. Where no confusion results, we occasionally treat this pair as a single formula, which we may gloss as “Exactly i ℓ s are m s.”

Evidently, the languages \mathcal{S}_0 , \mathcal{S}_1 and \mathcal{N} formalize the classical syllogistic, the syllogistic with unity, and the numerical syllogistic, respectively; similarly, \mathcal{S}_0^\dagger , \mathcal{S}_1^\dagger and \mathcal{N}^\dagger formalize their respective extended variants. Observe that the above semantics render formulas symmetric in their arguments: for example, $\mathfrak{A} \models \exists_{\leq i}(\ell, m)$ if and only if $\mathfrak{A} \models \exists_{\leq i}(m, \ell)$, and similarly for formulas featuring the quantifiers $\exists_{> i}$. Accordingly, we shall henceforth regard these arguments as unordered: that is, we identify the formulas $\exists_{\leq i}(\ell, m)$ and $\exists_{\leq i}(m, \ell)$, and similarly for $\exists_{> i}$. This will help to reduce notational clutter in some of the proofs.

If Θ is a set of formulas, we write $\mathfrak{A} \models \Theta$ if, for all $\theta \in \Theta$, $\mathfrak{A} \models \theta$. A formula θ is *satisfiable* if there exists a structure \mathfrak{A} such that $\mathfrak{A} \models \theta$; a set of formulas Θ is *satisfiable* if there exists \mathfrak{A} such that $\mathfrak{A} \models \Theta$. We call a formula of the form $\exists_{>i}(p, \bar{p})$ an *absurdity*, and use \perp to denote, indifferently, any absurdity. Evidently, \perp is unsatisfiable. If, for all structures \mathfrak{A} , $\mathfrak{A} \models \Theta$ implies $\mathfrak{A} \models \psi$, we say that Θ *entails* ψ , and write $\Theta \models \psi$. In the case where $\Theta = \{\theta\}$, we say that θ *entails* ψ . Thus, for example, the entailment

$$\{\exists_{\leq 1}(o, p), \exists_{\leq 1}(o, \bar{p}), \exists_{\leq 1}(q, \bar{o}), \exists_{> 1}(q, \bar{r})\} \models \exists_{\leq 1}(q, r). \quad (12)$$

formalizes the valid argument (9).

If $\varphi = \exists_{\leq i}(\ell, m)$, we write $\bar{\varphi}$ to denote $\exists_{>i}(\ell, m)$; and if $\varphi = \exists_{>i}(\ell, m)$, we write $\bar{\varphi}$ to denote $\exists_{\leq i}(\ell, m)$. Thus, $\bar{\bar{\varphi}} = \varphi$, and, in any structure \mathfrak{A} , $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \not\models \bar{\varphi}$. Informally, we may regard $\bar{\varphi}$ as the *negation* of φ . It will sometimes be convenient to restrict attention to formulas featuring only a limited selection of atoms. If $\mathbf{P}' \subseteq \mathbf{P}$, and \mathcal{L} is any of the languages \mathcal{S}_z or \mathcal{S}_z^\dagger , we denote the set of \mathcal{L} -formulas φ involving only atoms in \mathbf{P}' by $\mathcal{L}(\mathbf{P}')$. Since $\varphi \in \mathcal{L}(\mathbf{P}')$ evidently implies $\bar{\varphi} \in \mathcal{L}(\mathbf{P}')$, we may regard all these languages as *closed under negation*. We call a subset $\Phi \subseteq \mathcal{L}(\mathbf{P}')$ *complete for $\mathcal{L}(\mathbf{P}')$* if, for every $\varphi \in \mathcal{L}(\mathbf{P}')$, either $\varphi \in \Phi$ or $\bar{\varphi} \in \Phi$; reference to $\mathcal{L}(\mathbf{P}')$ is suppressed if clear from context.

Complete sets of formulas will play an important role in the sequel, and we employ the following abbreviations to help define them. Where the language (\mathcal{S}_z or \mathcal{S}_z^\dagger) is clear from context, and $0 \leq i \leq z$, we write $\exists_{\leq i}^*(\ell, m)$ for the set of formulas

$$\{\exists_{\leq i}(\ell, m), \dots, \exists_{\leq z}(\ell, m)\},$$

and $\exists_{>i}^*(\ell, m)$ for the set of formulas

$$\{\exists_{>0}(\ell, m), \dots, \exists_{>i}(\ell, m)\}.$$

In addition, for $0 < i \leq z$, we write $\exists_{=i}^*(\ell, m)$ for the set of formulas

$$\{\exists_{>0}(\ell, m), \dots, \exists_{>i-1}(\ell, m), \exists_{\leq i}(\ell, m), \dots, \exists_{\leq z}(\ell, m)\}.$$

(Thus, in the languages \mathcal{S}_1 and \mathcal{S}_1^\dagger , $\exists_{=1}^*(\ell, m)$ and $\exists_{=1}(\ell, m)$ coincide.) It is easy to see that, for any literals ℓ, m , any structure \mathfrak{A} , and any i ($0 \leq i \leq z$), $\mathfrak{A} \models \exists_{\leq i}^*(\ell, m)$ if and only if $\mathfrak{A} \models \exists_{\leq i}(\ell, m)$; similarly, $\mathfrak{A} \models \exists_{>i}^*(\ell, m)$ if and only if $\mathfrak{A} \models \exists_{>i}(\ell, m)$. In addition, for $0 < i \leq z$, $\mathfrak{A} \models \exists_{=i}^*(\ell, m)$ if and only if $\mathfrak{A} \models \exists_{=i}(\ell, m)$; moreover, exactly one of $\mathfrak{A} \models \exists_{\leq(i-1)}^*(\ell, m)$, $\mathfrak{A} \models \exists_{=i}^*(\ell, m)$ or $\mathfrak{A} \models \exists_{>i}^*(\ell, m)$ holds.

As mentioned above, the satisfiability problems for \mathcal{S}_0 and \mathcal{S}_0^\dagger —i.e. the classical syllogistic and the extended classical syllogistic—are both NLOGSPACE-complete. We end this section with a contrasting result on the complexity of satisfiability for \mathcal{S}_z and \mathcal{S}_z^\dagger , where $z > 0$.

Theorem 2.1. *For all $z > 0$, the problem of determining the satisfiability of a given set of \mathcal{S}_z -formulas is NP-TIME-complete, and similarly for \mathcal{S}_z^\dagger -formulas.*

Proof. Let Φ be any satisfiable set of \mathcal{S}_z^\dagger -formulas; we claim that Φ has a model over a domain of size at most $(z+1)|\Phi|$. Indeed, suppose $\mathfrak{A} \models \Phi$. For any formula $\exists_{>i}(\ell, m)$, select $(i+1)$ elements satisfying ℓ and m . Let B be the set of selected elements, and let \mathfrak{B} be the restriction of \mathfrak{A} to B . It is obvious that $\mathfrak{B} \models \Phi$, and $|B| \leq (z+1)|\Phi|$, proving the claim. Membership of the satisfiability problem for \mathcal{S}_z^\dagger in NPTime follows.

It remains to show NPTime-hardness of the satisfiability problem for \mathcal{S}_1 . Let \mathcal{T} be the language consisting of \mathcal{S}_1 together with formulas of the forms $\exists_{\leq 3}(p, p)$, where p is an atom. It is shown in Pratt-Hartmann (2008, Lemma 1) that the satisfiability problem for \mathcal{T} is NPTime-hard, using a straightforward reduction of graph-3-colourability. We need only reduce the satisfiability problem for \mathcal{T} to that for \mathcal{S}_1 . Let Φ be any set of \mathcal{T} -formulas, then. For any formula $\varphi = \exists_{\leq 3}(p, p)$, let o, o' be new atoms, and replace φ by the set of \mathcal{S}_1 -formulas $\{\exists_{\leq 1}(p, \bar{o}), \exists_{\leq 1}(o, o'), \exists_{\leq 1}(o, \bar{o}')\}$. Let the resulting set of \mathcal{S}_1 -formulas be Ψ . Evidently, Ψ entails every formula of Φ ; conversely, any structure \mathfrak{A} such that $\mathfrak{A} \models \Phi$ can easily be expanded to a structure \mathfrak{A}' such that $\mathfrak{A}' \models \Psi$. This completes the reduction. \square

We remark that, when considering the satisfiability problem for \mathcal{S}_z^\dagger , the integer z is a constant: thus, there is only a fixed number of quantifiers $\exists_{\leq i}$ or $\exists_{>i}$, so that we do not need to worry about the coding scheme (unary or binary) for the numerical subscripts. By contrast, for the language \mathcal{N}^\dagger , which features all these quantifiers, the coding of numerical subscripts is a significant issue. That the satisfiability problem for \mathcal{N}^\dagger remains in NPTime—even when numerical subscripts are coded as bit-strings—requires a clever combinatorial argument due to Eisenbrand and Shmonin (2006); for details, see Pratt-Hartmann (2008).

3 Syllogistic proof systems

Let \mathcal{L} be any of the languages \mathcal{S}_z or \mathcal{S}_z^\dagger ($z \geq 0$). A *sylogistic rule* in \mathcal{L} is a pair Θ/θ , where Θ is a finite set (possibly empty) of \mathcal{L} -formulas, and θ an \mathcal{L} -formula. We call Θ the *antecedents* of the rule, and θ its *consequent*. We generally display rules in ‘natural-deduction’ style. For example,

$$\frac{\exists_{\leq 0}(q, \bar{o}) \quad \exists_{>0}(p, q)}{\exists_{>0}(p, o)} \quad \frac{\exists_{\leq 0}(q, o) \quad \exists_{>0}(p, q)}{\exists_{>0}(p, \bar{o})}, \quad (13)$$

where p, q and o are atoms, are syllogistic rules in \mathcal{S}_0 (hence in all larger languages); they correspond to the traditional syllogisms *Darii* and *Ferio*, respectively:

Every q is an o	No q is an o
<u>Some p is a q</u>	<u>Some p is a q</u>
Some p is an o	Some p is not an o .

We call a syllogistic rule *sound* if its antecedents entail its consequent. Thus, the syllogistic rules (13) are sound. More generally, for all i, j, z ($0 \leq i \leq j \leq z$),

$$\frac{\exists_{\leq i}(q, \bar{o}) \quad \exists_{> j}(p, q)}{\exists_{> (j-i)}(p, o)} \quad \frac{\exists_{\leq i}(q, o) \quad \exists_{> j}(p, q)}{\exists_{> (j-i)}(p, \bar{o})}$$

are sound syllogistic rules in \mathcal{S}_z (and again in all larger languages).

Let X be a set of syllogistic rules in \mathcal{L} . A *substitution* is a function $g : \mathbf{P} \rightarrow \mathbf{P}$; we extend g to \mathcal{L} -formulas and to sets of \mathcal{L} -formulas in the obvious way. An *instance* of a syllogistic rule Θ/θ is the syllogistic rule $g(\Theta)/g(\theta)$, where g is a substitution. Denote by $\mathbb{P}(\mathcal{L})$ the set of subsets of \mathcal{L} . We define the *direct syllogistic derivation relation* \vdash_X to be the smallest relation on $\mathbb{P}(\mathcal{L}) \times \mathcal{L}$ satisfying:

1. if $\theta \in \Theta$, then $\Theta \vdash_X \theta$;
2. if $\{\theta_1, \dots, \theta_n\}/\theta$ is a syllogistic rule in X , g a substitution, $\Theta = \Theta_1 \cup \dots \cup \Theta_n$, and $\Theta_i \vdash_X g(\theta_i)$ for all i ($1 \leq i \leq n$), then $\Theta \vdash_X g(\theta)$.

Where the language \mathcal{L} is clear from context, we omit reference to it; further, we typically contract *syllogistic rule* to *rule*. Instances of the relation \vdash_X can always be established by *derivations* in the form of finite trees in the usual way. For instance, the derivation

$$\frac{\exists_{\leq 0}(o, r) \quad \frac{\exists_{\leq 0}(q, \bar{o}) \quad \exists_{> 0}(p, q)}{\exists_{> 0}(p, o)}}{\exists_{> 0}(p, \bar{r})}$$

establishes that, for any set of syllogistic rules X containing the rules (13),

$$\{\exists_{\leq 0}(o, r), \exists_{\leq 0}(q, \bar{o}), \exists_{> 0}(p, q)\} \vdash_X \exists_{> 0}(p, \bar{r}).$$

In the sequel, we reason freely about derivations in order to establish properties of derivation relations.

The derivation relation \vdash_X is said to be *sound* if $\Theta \vdash_X \theta$ implies $\Theta \models \theta$, and *complete* (for \mathcal{L}) if $\Theta \models \theta$ implies $\Theta \vdash_X \theta$. (Of course, this use of the word ‘complete’ is unrelated to the notion of a *complete* set of $\mathcal{L}(\mathbf{P}')$ -formulas defined in Sec. 2.) Evidently, it is the existence of sound and complete derivation relations that interest us, because they would yield convenient procedures for discovering entailments such as (12). A set Θ of formulas is *inconsistent* (with respect to \vdash_X) if $\Theta \vdash_X \perp$ for some absurdity \perp ; otherwise, *consistent*. It is obvious that, for any set of rules X , \vdash_X is sound if and only if every rule in X is sound.

In the sequel, we will need to consider a stronger notion of syllogistic derivation, incorporating a form of indirect reasoning. Let \mathcal{L} be one of the languages considered above, and X a set of syllogistic rules in \mathcal{L} . We define the *indirect syllogistic derivation relation* \Vdash_X to be the smallest relation on $\mathbb{P}(\mathcal{L}) \times \mathcal{L}$ satisfying:

1. if $\theta \in \Theta$, then $\Theta \Vdash_X \theta$;
2. if $\{\theta_1, \dots, \theta_n\}/\theta$ is a syllogistic rule in X , g a substitution, $\Theta = \Theta_1 \cup \dots \cup \Theta_n$, and $\Theta_i \Vdash_X g(\theta_i)$ for all i ($1 \leq i \leq n$), then $\Theta \Vdash_X g(\theta)$.
3. if $\Theta \cup \{\theta\} \Vdash_X \perp$, where \perp is any absurdity, then $\Theta \Vdash_X \bar{\theta}$.

The only difference is the addition of the final clause, which allows us to derive a formula $\bar{\theta}$ from premises Θ if we can derive an absurdity from Θ together with θ . Instances of the indirect derivation relation \Vdash_X may also be established by constructing derivations, except that we need a little more machinery to keep track of premises. This may be done as follows. Suppose we have a derivation (direct or indirect) showing that $\Theta \cup \{\theta\} \Vdash_X \perp$, for some absurdity \perp . Let this derivation be displayed as

$$\begin{array}{ccccccc} \theta_1 & \cdots & \theta_n & \theta & \cdots & \theta & \\ & & \vdots & & & & \\ & & \perp, & & & & \end{array}$$

where $\theta_1, \dots, \theta_n$ is a list of formulas of Θ (not necessarily exhaustive, and with repeats allowed). Applying Clause 3 of the definition of \Vdash_X , we have $\Theta \Vdash_X \bar{\theta}$, which we take to be established by the derivation

$$\begin{array}{ccccccc} \theta_1 & \cdots & \theta_n & [\theta]^1 & \cdots & [\theta]^1 & \\ & & \vdots & & & & \\ & & \perp & & & & \\ \hline & & \bar{\theta} & & & & \end{array} \text{ (RAA)}^1.$$

The tag (RAA) stands for *reductio ad absurdum*; the square brackets indicate that the enclosed instances of θ have been *discharged*, i.e. no longer count among the premises; and the numerical indexing is simply to make the derivation history clear. Note that there is nothing to prevent θ from occurring among the $\theta_1, \dots, \theta_n$; that is to say, we do not have to discharge all (or indeed any) instances of the premise θ if we do not want to.

The notions of *soundness* and *completeness* are defined for indirect derivation relations in exactly the same way as for direct derivation relations. Again, it should be obvious that, for any set of rules X , \Vdash_X is sound if and only if every rule in X is sound. It is important to understand that *reductio ad absurdum* cannot be formulated as a syllogistic rule in the technical sense defined here; rather, it is part of the proof-theoretic machinery that converts any set of rules X into the derivation relation \Vdash_X . It is shown in Pratt-Hartmann and Moss (2009) that, for both the classical syllogistic, \mathcal{S}_0 , and its extension \mathcal{S}_0^\dagger , there exist finite sets of rules X such that the *direct* derivation relation \vdash_X is sound and complete. (That is: *reductio ad absurdum* is not needed.) However, the same paper considers various extensions of the classical syllogistic for which there are sound and complete *indirect* syllogistic derivation relations, but no sound and complete *direct* ones. Thus, it is in general important to distinguish

these two kinds of proof systems. We show below that neither \mathcal{S}_z nor \mathcal{S}_z^\dagger has even a sound and complete *indirect* syllogistic proof system, for all $z > 0$.

We end this section with two simple results on syllogistic derivations.

Lemma 3.1. *Let \mathcal{L} be any of the languages \mathcal{S}_z or \mathcal{S}_z^\dagger ($z \geq 0$), X a set of syllogistic rules in \mathcal{L} , and \mathbf{P}' a non-empty subset of \mathbf{P} . Let $\theta \in \mathcal{L}(\mathbf{P}')$ and $\Theta \subseteq \mathcal{L}(\mathbf{P}')$. If there is a derivation (direct or indirect) of θ from Θ using X , then there is such a derivation involving only the atoms of \mathbf{P}' . Further, if there is a derivation of an absurdity from Θ , then there is a derivation of an absurdity \perp from Θ such that $\perp \in \mathcal{L}(\mathbf{P}')$.*

Proof. Given a derivation of θ from Θ , uniformly replace any atom not in \mathbf{P}' with one that is. Similarly for the second statement. \square

Lemma 3.2. *Let \mathcal{L} be any of the languages \mathcal{S}_z or \mathcal{S}_z^\dagger ($z \geq 0$), and X a set of syllogistic rules in \mathcal{L} . Let $\mathbf{P}' \subseteq \mathbf{P}$ be non-empty, and Ψ a complete set of $\mathcal{L}(\mathbf{P}')$ -formulas. If $\Psi \Vdash_{\mathsf{X}} \perp$, then $\Psi \vdash_{\mathsf{X}} \perp$.*

Proof. Suppose that there is an indirect derivation of some absurdity \perp from Ψ , using the rules X . By Lemma 3.1, we may assume all formulas involved are in $\mathcal{L}(\mathbf{P}')$. Let the number of applications of (RAA) employed in this derivation be k ; and assume without loss of generality that \perp is chosen so that this number k is minimal. If $k > 0$, consider the last application of (RAA) in this derivation, which derives a formula, say, $\bar{\psi}$, discharging a premise ψ . Then there is an (indirect) derivation of some absurdity \perp' from $\Psi \cup \{\psi\}$, employing fewer than k applications of (RAA). By minimality of k , $\psi \notin \Psi$, and so, by the completeness of Ψ , $\bar{\psi} \in \Psi$. But then we can replace our original derivation of $\bar{\psi}$ with the trivial derivation, so obtaining a derivation of \perp from Ψ with fewer than k applications of (RAA), a contradiction. Therefore, $k = 0$, or, in other words, $\Psi \vdash_{\mathsf{X}} \perp$. \square

4 No sound and complete syllogistic systems for \mathcal{S}_z or \mathcal{S}_z^\dagger

In this section, we prove that none of the languages \mathcal{S}_z or \mathcal{S}_z^\dagger ($z > 0$) has a sound and complete indirect syllogistic proof system. The strategy we adopt is identical to that employed in Pratt-Hartmann (2009) to obtain analogous results for the languages \mathcal{N} and \mathcal{N}^\dagger . However, the specific construction required to cope with the restriction to \mathcal{S}_z and \mathcal{S}_z^\dagger is new, and more involved.

To reduce clutter in the proof, we begin with the most interesting case: $z = 1$.

Theorem 4.1. *There is no finite set X of syllogistic rules in either \mathcal{S}_1 or \mathcal{S}_1^\dagger such that \Vdash_{X} is sound and complete.*

Proof. We first prove the result for \mathcal{S}_1^\dagger ; the result for \mathcal{S}_1 will then follow by an easy adaptation. Henceforth, then, let X be a finite set of sound syllogistic rules in \mathcal{S}_1^\dagger . We show that \Vdash_{X} is not complete.

Let $n \geq 4$, and let \mathbf{P}^n be a subset of \mathbf{P} of cardinality $4n + 2$: we shall write $\mathbf{P}^n = \{p_0, \dots, p_{2n-1}, q_0, \dots, q_{2n+1}\}$. Now let Γ^n be the set of $\mathcal{S}_1^1(\mathbf{P}^n)$ -formulas given in (14)–(34), which, for perspicuity, we divide into groups. (Recall that the arguments of formulas are taken to be *unordered*.) The first group concerns the literals p_i only:

$$\exists_{=1}(p_i, p_{i+1}) \quad (0 \leq i \leq 2n - 2) \quad (14)$$

$$\exists_{=1}(p_i, p_{i+3}) \quad (i \text{ even}) \wedge (0 \leq i \leq 2n - 4) \quad (15)$$

$$\exists_{\leq 0}^*(p_0, p_{2n-1}) \quad (16)$$

$$\begin{aligned} \exists_{>1}^*(p_i, p_j) \quad & (0 \leq i \leq j \leq 2n - 1) \wedge (j \neq i + 1) \wedge \\ & (i \text{ odd} \vee j \neq i + 3) \wedge (i \neq 0 \vee j \neq 2n - 1) \end{aligned} \quad (17)$$

$$\exists_{\leq 0}^*(p_i, \bar{p}_i) \quad (0 \leq i \leq 2n - 1) \quad (18)$$

$$\exists_{>1}^*(p_i, \bar{p}_j) \quad (0 \leq i \leq 2n - 1) \wedge (0 \leq j \leq 2n - 1) \wedge (i \neq j) \quad (19)$$

$$\exists_{>1}^*(\bar{p}_i, \bar{p}_j) \quad (0 \leq i \leq j \leq 2n - 1). \quad (20)$$

The second group concerns the literals q_i only:

$$\exists_{=1}(q_i, q_{i+1}) \quad (i \text{ even}) \wedge (0 \leq i \leq 2n) \quad (21)$$

$$\begin{aligned} \exists_{>1}^*(q_i, q_j) \quad & (0 \leq i \leq j \leq 2n + 1) \wedge \\ & (i \text{ odd} \vee j \neq i + 1) \end{aligned} \quad (22)$$

$$\exists_{\leq 0}^*(q_i, \bar{q}_i) \quad (0 \leq i \leq 2n + 1) \quad (23)$$

$$\exists_{>1}^*(q_i, \bar{q}_j) \quad (0 \leq i \leq 2n + 1) \wedge (0 \leq j \leq 2n + 1) \wedge (i \neq j) \quad (24)$$

$$\exists_{>1}^*(\bar{q}_i, \bar{q}_j) \quad (0 \leq i \leq j \leq 2n + 1). \quad (25)$$

The third group mixes the literals p_i and q_i :

$$\exists_{=1}(p_{i+1}, q_i) \quad (i \text{ even}) \wedge (0 \leq i \leq 2n - 2) \quad (26)$$

$$\exists_{=1}(p_i, q_{i+1}) \quad (0 \leq i \leq 2n - 1) \quad (27)$$

$$\exists_{=1}(p_i, q_{i+3}) \quad (i \text{ even} \wedge 0 \leq i \leq 2n - 2) \quad (28)$$

$$\begin{aligned} \exists_{>1}^*(p_i, q_j) \quad & (0 \leq i \leq 2n - 1) \wedge (0 \leq j \leq 2n + 1) \wedge (j \neq i + 1) \wedge \\ & (i \text{ odd} \vee j \neq i + 3) \wedge (j \text{ odd} \vee i \neq j + 1) \end{aligned} \quad (29)$$

$$\exists_{\leq 0}^*(p_i, \bar{q}_i) \quad (0 \leq i \leq 2n - 1) \quad (30)$$

$$\exists_{\leq 0}^*(p_i, \bar{q}_{i+2}) \quad (0 \leq i \leq 2n - 1) \quad (31)$$

$$\begin{aligned} \exists_{>1}^*(p_i, \bar{q}_j) \quad & (0 \leq i \leq 2n - 1) \wedge (0 \leq j \leq 2n + 1) \wedge (j \neq i) \wedge \\ & (j \neq i + 2) \end{aligned} \quad (32)$$

$$\exists_{>1}^*(\bar{p}_i, q_j) \quad (0 \leq i \leq 2n - 1) \wedge (0 \leq j \leq 2n + 1) \quad (33)$$

$$\exists_{>1}^*(\bar{p}_i, \bar{q}_j) \quad (0 \leq i \leq 2n - 1) \wedge (0 \leq j \leq 2n + 1) \quad (34)$$

In fact, the formulas that will be doing most of the work here are (14), (16), (21), (30) and (31). The others are required only to ensure that we have a complete set of formulas.

Claim 4.2. Γ^n is a complete set of $\mathcal{S}_1^\dagger(\mathbf{P}^n)$ -formulas. On the other hand, Γ^n contains no absurdities.

Proof. Consider first any formula of $\mathcal{S}_1^\dagger(\mathbf{P}^n)$ whose arguments are $\{p_i, p_j\}$ ($0 \leq i \leq j \leq n-1$). Since the conditions on i and j in (14)–(17) are clearly exhaustive (taking account of the fact that the arguments of formulas are unordered), we see that Γ^n contains one of $\exists_{\leq 0}(p_i, p_j)$ or $\exists_{>0}(p_i, p_j)$, and one of $\exists_{\leq 1}(p_i, p_j)$ or $\exists_{>1}(p_i, p_j)$. The other argument-patterns are dealt with similarly. The second part of the lemma is ensured by the condition ($i \neq j$) in the sets of formulas (19) and (24). \square

Claim 4.3. Γ^n is unsatisfiable.

Proof. Suppose $\mathfrak{A} \models \Gamma^n$. From (14), let a_h satisfy p_{2h} and p_{2h+1} ($0 \leq h \leq n-1$). From (30) and (31), a_h also satisfies $q_{2h}, q_{2h+1}, q_{2h+2}$ and q_{2h+3} . Hence, from (21), we have $a_0 = a_1 = \dots = a_{n-1}$. But then this common element satisfies p_0 and p_{2n-1} , contradicting (16). \square

We now proceed to define a collection of satisfiable variants of Γ_t^n . For all t ($1 \leq t \leq n-2$), let

$$\Gamma_t^n = (\Gamma^n \setminus \{\exists_{>0}(p_{2t-1}, p_{2t}), \exists_{>0}(p_{2t-2}, p_{2t+1}), \exists_{\leq 1}(q_{2t}, q_{2t+1})\}) \cup \{\exists_{\leq 0}(p_{2t-1}, p_{2t}), \exists_{\leq 0}(p_{2t-2}, p_{2t+1}), \exists_{>1}(q_{2t}, q_{2t+1})\}.$$

Since Γ^n is complete, so is Γ_t^n . The difference between Γ^n and Γ_t^n that will be doing most of the work here is that the latter set lacks the formula $\exists_{\leq 1}(q_{2t}, q_{2t+1})$. This disrupts the argument of Claim 4.3: the best we can now infer is that $a_0 = a_1 = \dots = a_{t-1}$, and $a_t = a_{t+1} = \dots = a_{n-1}$, so that there need be no element satisfying both p_0 and p_{2n-1} .

Claim 4.4. If $1 \leq t < t' \leq n-2$, then $\Gamma_t^n \cap \Gamma_{t'}^n \subseteq \Gamma^n$.

Proof. Straightforward check. \square

To show that Γ_t^n is satisfiable, we define a structure \mathfrak{B}_t^n as follows. The domain of \mathfrak{B}_t^n consists of elements $a, a', b_{i,j}, b'_{i,j}, c_{i,j}, c'_{i,j}, d_{i,j}, d'_{i,j}, e$ and e' , with indices subject to the conditions in the middle column of Table 1 (interpreted conjunctively); the atoms satisfied by these elements in \mathfrak{B}_t^n are listed in the right-most column of Table 1. Note that the elements e and e' satisfy no atoms at all. Roughly, the elements a and a' ensure the truth of formulas in Γ_t^n of the form $\exists_{>0}(\ell, m)$ for which the corresponding $\exists_{\leq 1}(\ell, m)$ is *also* in Γ_t^n (i.e. a uniqueness claim), while the remaining elements ensure the truth of formulas in Γ_t^n of the form $\exists_{>1}(\ell, m)$. The main task in the proof of Claim 4.5 below is to ensure that these latter elements do not spoil any uniqueness claims.

Claim 4.5. For all $n \geq 4$ and all t ($1 \leq t \leq n-2$), $\mathfrak{B}_t^n \models \Gamma_t^n$.

Proof. We consider the formulas (14)–(34) in turn, taking account of the differences between Γ^n and Γ_t^n as we encounter them.

Element name	index conditions	atoms satisfied
a		$p_0, \dots, p_{2t-1}, q_0, \dots, q_{2t+1},$
a'		$p_{2t}, \dots, p_{2n-1}, q_{2t}, \dots, q_{2n+1},$
$b_{i,j}, b'_{i,j}$	$(0 \leq i \leq j \leq 2n-1)$ $(j \neq i+1)$ $(i \text{ odd} \vee j \neq i+3)$ $(i \neq 0 \vee j \neq 2n-1)$	$p_i, q_i, q_{i+2}, p_j, q_j, q_{j+2}$
$c_{i,j}, c'_{i,j}$	$(0 \leq i \leq 2n-1)$ $(0 \leq j \leq 2n+1)$ $(j \neq i+1)$ $(i \text{ odd} \vee j \neq i+3)$ $(j \text{ odd} \vee i \neq j+1)$	p_i, q_i, q_{i+2}, q_j
$d_{i,j}, d'_{i,j}$	$(0 \leq i \leq j \leq 2n+1)$ $(i \text{ odd} \vee j \neq i+1)$	q_i, q_j
e, e'		

Table 1: Definition of the model \mathfrak{B}_t^n : elements $b_{i,j}$, $b'_{i,j}$, $c_{i,j}$, $c'_{i,j}$, $d_{i,j}$ and $d'_{i,j}$ exist only for those pairs of indices i, j satisfying *all* the indicated index conditions.

- (14): For $0 \leq i \leq 2t-2$, a satisfies the atoms p_i and p_{i+1} , whereas a' does not; for $2t \leq i \leq 2n-2$, a' satisfies the atoms p_i and p_{i+1} , whereas a does not; for $i = 2t-1$, neither a nor a' satisfies (both) these atoms, but then Γ_t^n replaces $\exists_{>0}(p_{2t-1}, p_{2t})$ by $\exists_{\leq 0}(p_{2t-1}, p_{2t})$. It remains to check that no other element satisfies these atoms. The only danger is from $b_{i',j'}$ and $b'_{i',j'}$, where $i' = i$ and $j' = i+1$; but the condition $j' \neq i'+1$ (middle column of Table 1) rules this combination of indices out.
- (15): Almost identical to the argument for (14).
- (16): Almost identical to the argument for (14).
- (17): The elements $b_{i,j}$ and $b_{i',j'}$ both satisfy the atoms p_i and p_j . Notice that the conditions on the indices in (17) are matched by the relevant conditions in the middle column of Table 1, so that all formulas are accounted for.
- (18): Trivially satisfied.
- (19): If $i \neq j$, then both $c_{i,i}$ and $c'_{i,i}$, which exist for all i ($0 \leq i \leq 2n-1$), satisfy the literals p_i , and \bar{p}_j .
- (20): Both e and e' satisfy the literals \bar{p}_i and \bar{p}_j .
- (21): For $0 \leq i \leq 2t-2$, a satisfies the atoms q_i and q_{i+1} , whereas a' does not; for $2t+2 \leq i \leq 2n$, a' satisfies the atoms q_i and q_{i+1} , whereas a does not; for $i = 2t$, both a and a' satisfy (both) these atoms, but then Γ_t^n replaces $\exists_{\leq 1}(q_{2t}, q_{2t+1})$ by $\exists_{>1}(q_{2t}, q_{2t+1})$. It remains to check that no

other element satisfies these atoms. Considering the right-hand column of Table 1, the only danger is from: (i) $b_{i',j'}$ and $b'_{i',j'}$, where $i' = i$ and $j' = i + 1$; (ii) $b_{i',j'}$ and $b'_{i',j'}$, where $i' + 2 = i$ and $j' = i + 1$ (with i' even); (iii) $b_{i',j'}$ and $b'_{i',j'}$, where $i' + 2 = i$ and $j' + 2 = i + 1$; (iv) $b_{i',j'}$ and $b'_{i',j'}$, where $i' + 2 = i + 1$ and $j' = i$; (v) $c_{i',j'}$ and $c'_{i',j'}$, where $i' = i$ and $j' = i + 1$; (vi) $c_{i',j'}$ and $c'_{i',j'}$, where $i' + 2 = i$ and $j' = i + 1$ (with i' even); (vii) $c_{i',j'}$ and $c'_{i',j'}$, where $j' = i$ and $i' = i + 1$ (with j' even); (viii) $c_{i',j'}$ and $c'_{i',j'}$, where $i' + 2 = i + 1$ and $j' = i$; (ix) $d_{i',j'}$ and $d'_{i',j'}$, where $i' = i$ and $j' = i + 1$ (with i' even). However, the conditions in the middle column of Table 1 rule these combinations of indices out.

- (22): The elements $d_{i,j}$ and $d_{i',j'}$ both satisfy the atoms q_i and q_j . Notice that the conditions on the indices in (22) are matched by the relevant conditions in the middle column of Table 1, so that all formulas are accounted for.
- (23): Trivially satisfied.
- (24): If $i \neq j$, then both $d_{i,i}$ and $d'_{i,i}$, which exist for all i ($0 \leq i \leq 2n + 1$), satisfy the literals q_i , and \bar{q}_j .
- (25): Both o and o' satisfy the literals \bar{q}_i and \bar{q}_j .
- (26): For $0 \leq i \leq 2t - 2$, a satisfies the atoms p_{i+1} and q_i , whereas a' does not; for $2t \leq i \leq 2n - 2$, a' satisfies the atoms p_{i+1} and q_i , whereas a does not. We check that no other element satisfies these atoms. Considering the right-hand column of Table 1, the only danger is from: (i) $b_{i',j'}$ and $b'_{i',j'}$, where $i' = i$ and $j' = i + 1$; (ii) $b_{i',j'}$ and $b'_{i',j'}$, where $i' + 2 = i$ and $j' = i + 1$ (with i' even); (iii) $c_{i',j'}$ and $c'_{i',j'}$, where $i' = i + 1$ and $j' = i$ (with j' even). However, the conditions in the middle column of Table 1 rule these combinations of indices out.
- (27): For $0 \leq i \leq 2t - 1$, a satisfies the atoms p_i and q_{i+1} , whereas a' does not; for $2t \leq i \leq 2n - 1$, a' satisfies the atoms p_i and q_{i+1} , whereas a does not. We check that no other element satisfies these atoms. Considering the right-hand column of Table 1, the only danger is from: (i) $b_{i',j'}$ and $b'_{i',j'}$, where $i' = i$ and $j' = i + 1$; (ii) $b_{i',j'}$ and $b'_{i',j'}$, where $i' + 2 = i + 1$ and $j' = i$; (iii) $c_{i',j'}$ and $c'_{i',j'}$, where $i' = i$ and $j' = i + 1$. However, the conditions in the middle column of Table 1 rule these combinations of indices out.
- (28): For $0 \leq i \leq 2t - 2$, a satisfies the atoms p_i and q_{i+3} , whereas a' does not; for $2t \leq i \leq 2n - 2$, a' satisfies the atoms p_i and q_{i+3} , whereas a does not. We check that no other element satisfies these atoms. Considering the right-hand column of Table 1, the only danger is from: (i) $b_{i',j'}$ and $b'_{i',j'}$, where $i' = i$ and $j' = i + 3$ (with i' even); (ii) $b_{i',j'}$ and $b'_{i',j'}$, where $i' = i$ and $j' + 2 = i + 3$; (iii) $c_{i',j'}$ and $c'_{i',j'}$, where $i' = i$ and $j' = i + 3$ (with i' even). However, the conditions in the middle column of Table 1 rule these combinations of indices out.

- (29): The elements $c_{i,j}$ and $c'_{i',j'}$ both satisfy the atoms p_i and q_j . Notice that the conditions on the indices in (29) are matched by the relevant conditions in the middle column of Table 1, so that all formulas are accounted for.
- (30): It is readily checked that every element satisfying p_i also satisfies q_i .
- (31): It is readily checked that every element satisfying p_i also satisfies q_{i+2} .
- (32): The elements $c_{i,i}$ and $c'_{i,i}$ both satisfy the literals p_i and \bar{q}_j , as long as $j \neq i$ and $j \neq i+2$. Note that these elements exist for all i ($0 \leq i \leq 2n-1$), so all the relevant formulas are accounted for.
- (33): The elements $d_{j,j}$ and $d'_{j,j}$ both satisfy the literals \bar{p}_i and q_j . Note that these elements exist for all j ($0 \leq j \leq 2n+1$), so all the relevant formulas are accounted for.
- (25): Both e and e' satisfy the literals \bar{p}_i and \bar{q}_j .

□

The key step in the proof is to show that, for any finite set of sound syllogistic rules, we can make n sufficiently large that those rules cannot be used to infer anything new from Γ^n . For suppose \mathbf{X} is a finite set of sound rules in \mathcal{S}_1^\dagger . Let r be the maximum number of premises in any rule in \mathbf{X} , and let $n \geq r+4$.

Claim 4.6. *If $\Gamma^n \vdash_{\mathbf{X}} \theta$, then $\theta \in \Gamma_n$.*

Proof. Consider any derivation establishing that $\Gamma^n \vdash_{\mathbf{X}} \theta$. By Lemma 3.1, we may assume that that derivation features only atoms in \mathbf{P}^n . We show by induction on the number of steps (proof-rule instances) in the derivation that $\theta \in \Gamma^n$. If there are no steps, then $\theta \in \Gamma_n$ by definition. Otherwise, consider the last proof rule instance, and let its antecedents be Θ . Thus, $|\Theta| \leq n-4$; moreover, since the elements of Θ have shorter derivations than θ , $\Theta \subseteq \Gamma^n$, by inductive hypothesis. Now consider the formulas

$$\exists_{>0}(p_{2h-1}, p_{2h}), \quad \exists_{>0}(p_{2h-2}, p_{2h+1}), \quad \exists_{\leq 1}(q_{2h}, q_{2h+1}),$$

where $1 \leq h \leq n-2$; and let us arrange these formulas on a rectangular grid, thus:

$h = 1$	$h = 2$	\dots	$h = n-2$
$\exists_{>0}(p_1, p_2)$	$\exists_{>0}(p_3, p_4)$	\dots	$\exists_{>0}(p_{2n-5}, p_{2n-4})$
$\exists_{>0}(p_0, p_3)$	$\exists_{>0}(p_2, p_5)$	\dots	$\exists_{>0}(p_{2n-6}, p_{2n-3})$
$\exists_{\leq 1}(q_2, q_3)$	$\exists_{\leq 1}(q_4, q_5)$	\dots	$\exists_{\leq 1}(q_{2n-4}, q_{2n-3})$

Since $|\Theta| \leq n-4$, we can find two columns in this grid which do not intersect Θ : in other words, there exist two values of h ($1 \leq h \leq n-2$) such that $\Theta \subseteq \Gamma_h^n$. For these values of h , $\mathfrak{B}_h^n \models \Theta$ by Claim 4.5, and hence $\mathfrak{B}_h^n \models \theta$, by the supposed soundness of the rules in \mathbf{X} . By the completeness of Γ_h^n , $\theta \in \Gamma_h^n$, whence $\theta \in \Gamma^n$, by Claim 4.4. □

We now complete the proof of the theorem. Pick any absurdity \perp . By Claim 4.2, $\perp \notin \Gamma^n$, and so, by Claim 4.6, $\Gamma^n \not\models_{\mathbf{X}} \perp$. By Lemma 3.2, $\Gamma^n \not\models_{\mathbf{X}} \perp$. Yet, $\Gamma^n \models \perp$, by Claim 4.3. Thus, $\models_{\mathbf{X}}$ is not complete, as required.

For \mathcal{S}_1 , simply delete from Γ^n and Γ_t^n all formulas not in that language, and define \mathfrak{B}_t^n as before. Since the formulas (14), (16), (21), (30) and (31) featuring in the proof of Claim 4.3 are all in \mathcal{S}_1 , and Γ^n and Γ_t^n also differ only in respect of \mathcal{S}_1 -formulas, the proof proceeds as for \mathcal{S}_1^\dagger . \square

Corollary 4.7. *For all $z \geq 1$, there is no finite set \mathbf{X} of syllogistic rules in either \mathcal{S}_z or \mathcal{S}_z^\dagger such that $\models_{\mathbf{X}}$ is sound and complete.*

Proof. For $z > 1$, we make the following changes to the proof of Theorem 4.1.

(i) In the definition of Γ^n , all occurrences of $\exists_{>1}^*$ are replaced by $\exists_{>z}^*$; and all occurrences of $\exists_{=1}$ are replaced by $\exists_{=1}^*$. (ii) The definition of Γ_t^n in terms of Γ^n is unaffected, namely:

$$\Gamma_t^n = (\Gamma^n \setminus \{\exists_{>0}(p_{2t-1}, p_{2t}), \exists_{>0}(p_{2t-2}, p_{2t+1}), \exists_{\leq 1}(q_{2t}, q_{2t+1})\}) \cup \{\exists_{\leq 0}(p_{2t-1}, p_{2t}), \exists_{\leq 0}(p_{2t-2}, p_{2t+1}), \exists_{>1}(q_{2t}, q_{2t+1})\}.$$

Thus, Γ_t^n contains (more precisely: *includes*) $\exists_{\leq 0}^*(p_{2t-1}, p_{2t})$, $\exists_{\leq 0}^*(p_{2t-2}, p_{2t+1})$ and $\exists_{=2}^*(q_{2t}, q_{2t+1})$. (iii) In the definition of \mathfrak{B}_t^n , instead of taking just two elements, $b_{i,j}$ and $b'_{i,j}$, we instead take $(z+1)$ elements, $b_{i,j}, b'_{i,j}, \dots, b'_{i,j}{}^{(z)}$; and similarly with $c_{i,j}$ and $d_{i,j}$. The argument then proceeds exactly as for Theorem 4.1. \square

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